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Hausdorff dimension of unique beta expansions

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Abstract

Given an integer $N \geqslant 2$ and a real number $\beta > 1$, let $\Gamma_{\beta,N}$ be the set of all $x = \sum_{i=1}^{\infty} d_i/\beta^i$ with $d_i \in \{0, 1, \cdots, N-1\}$ for all $i \geqslant 1$. The infinite sequence (d_i) is called a β-expansion of x. Let $\boldsymbol{U}_{\beta,N}$ be the set of all x's in $\Gamma_{\beta,N}$ which have unique β-expansions. We give explicit formula of the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$ for β in any admissible interval $[\beta_L, \beta_U]$, where β_L is a purely Parry number while β_U is a transcendental number whose quasi-greedy expansion of 1 is related to the classical Thue–Morse sequence. This allows us to calculate the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$ for almost every $\beta > 1$. In particular, this improves the main results of Gábor Kallós (1999, 2001). Moreover, we find that the dimension function $f(\beta) = \dim_H \boldsymbol{U}_{\beta,N}$ fluctuates frequently for $\beta \in (1, N)$.

Keywords: unique beta expansion, Hausdorff dimension, generalized Thue–Morse sequence, admissible block, admissible interval, transcendental number
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(Some figures may appear in colour only in the online journal)

1. Introduction

Given an integer $N \geqslant 2$ and a real number $\beta > 1$, we call the infinite sequence (d_i) a *β-expansion* of x if we can write

$$x = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i}$$

with $d_i \in \{0, 1, \cdots, N-1\}$ for all $i \geqslant 1$. Let $\Gamma_{\beta,N}$ be the set of all such x's, i.e.,

$$\Gamma_{\beta,N} = \left\{ \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} : d_i \in \{0, 1, \cdots, N-1\}, i \geqslant 1 \right\}.$$

Then $\Gamma_{\beta,N}$ is a self-similar set generated by the *iterated function systems* (IFS) $\{f_d(x) = (x+d)/\beta : d \in \{0, 1, \cdots, N-1\}\}$ (see [16]). Let $\{0, 1, \cdots, N-1\}^\infty$ be the set of all expansions (d_i) with each digit $d_i \in \{0, 1, \cdots, N-1\}$. We define the projection map Π_β from $\{0, 1, \cdots, N-1\}^\infty$ to $\Gamma_{\beta,N}$ by

$$\Pi_\beta((d_i)) = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i}. \tag{1}$$

When $\beta > N$, the IFS $\{f_d(\cdot) : d \in \{0, 1, \cdots, N-1\}\}$ satisfies the *strong separation condition* (SSC), and then the map Π_β is bijective which implies that every point in $\Gamma_{\beta,N}$ has a unique β-expansion. When $\beta = N$, the IFS $\{f_d(\cdot) : d \in \{0, 1, \cdots, N-1\}\}$ fails the SSC but satisfies the *open set condition* (OSC). Then all except for countably many points in $\Gamma_{\beta,N}$ have unique β-expansions.

However, when $\beta < N$, the IFS $\{f_d(\cdot) : d \in \{0, 1, \cdots, N-1\}\}$ fails the OSC. In this case, $\Gamma_{\beta,N} = [0, (N-1)/(\beta-1)]$ and almost every point in $\Gamma_{\beta,N}$ have continuum of β-expansions (see [9, 33, 35]). This has close connections to representations of real numbers in non-integer bases. After the seminal works of Rényi [31] and Parry [29] β-expansions were widely considered from many aspects of mathematics, such as dynamical systems, measure theory, probability, number theory and so on (see [10, 13, 15, 17, 30, 32, 34, 36]).

In 1990 Erdös, Joó and Komornik [15] showed for $N = 2$ that for $\beta \in (1, G)$ any internal point of $\Gamma_{\beta,N}$ has continuum of β-expansions, and for $\beta \in (G, 2)$ there exist infinitely many points of $\Gamma_{\beta,N}$ having unique β-expansions (see [18]), where $G = (1+\sqrt{5})/2$ is the golden ratio. Recently, Baker [7] generalized their result and showed for $N \geqslant 2$ that there exists $G_N \in (1, N)$ defined by

$$G_N = \begin{cases} k+1 & \text{if } N = 2k+1, \\ \dfrac{k + \sqrt{k^2+4k}}{2} & \text{if } N = 2k, \end{cases} \tag{2}$$

such that for each $\beta \in (1, G_N)$ any internal point of $\Gamma_{\beta,N}$ has continuum of β-expansions, and for $\beta \in (G_N, N)$ there exist infinitely many points in $\Gamma_{\beta,N}$ having unique β-expansions (see [25]).

Let $\boldsymbol{U}_{\beta,N}$ be the set of all x's in $\Gamma_{\beta,N}$ which have unique β-expansions, i.e. for any $x \in \boldsymbol{U}_{\beta,N}$ there exists a unique sequence $(d_i) \in \{0, 1, \cdots, N-1\}^\infty$ such that $x = \sum_{i=1}^\infty d_i/\beta^i$. When $\beta \in (1, N)$, the structure of $\boldsymbol{U}_{\beta,N}$ is complex (see [11–13, 18, 20, 21, 24]). Recently, De Vries and Komornik [13] showed that there exists $\beta_c(N) \in (G_N, N)$ such that (see also [14, 18, 25])

- if $\beta \in (G_N, \beta_c(N))$, then $|\boldsymbol{U}_{\beta,N}| = \aleph_0$;
- if $\beta = \beta_c(N)$, then $\dim_H \boldsymbol{U}_{\beta,N} = 0$ but $|\boldsymbol{U}_{\beta,N}| = 2^{\aleph_0}$;
- if $\beta \in (\beta_c(N), N)$, then $0 < \dim_H \boldsymbol{U}_{\beta,N} < 1$.

Here $\beta_c(N)$ is the Komornik–Loreti constant defined as the unique positive solution of the equation $1 = \sum_{i=1}^\infty \lambda_i/\beta^i$, where $(\lambda_i) = (\lambda_i(N))$ is given by (see [23])

$$\lambda_i(N) = \begin{cases} k - 1 + \tau_i & \text{if } N = 2k, \\ k + \tau_i - \tau_{i-1} & \text{if } N = 2k+1, \end{cases} \tag{3}$$

with $(\tau_i)_{i=0}^\infty$ the classical Thue–Morse sequence starting at (see [4])

$$0110\,1001\,1001\,0110 \cdots.$$

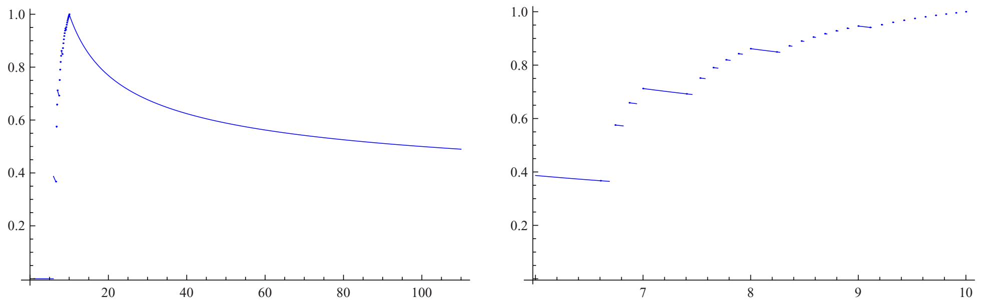

Figure 1. The Hausdorff dimension of $\boldsymbol{U}_{\beta,10}$ for $\beta \in (1, 110)$ (left column), and for β in the one-level and two-level admissible intervals in $(\beta_c(10), 10) \approx (5.976, 10)$ (right column).

Allouche and Cosnard [2] showed that $\beta_c(2)$ is a transcendental number. Later, Komornik and Loreti [23] showed that $\beta_c(N)$ is transcendental for any $N \geqslant 2$.

The purpose of this paper is to investigate the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$. From the above observation it follows that

- if $\beta \in (1, \beta_c(N)]$, then $\dim_H \boldsymbol{U}_{\beta,N} = 0$;
- if $\beta \in [N, \infty)$, then $\dim_H \boldsymbol{U}_{\beta,N} = \dim_H \Gamma_{\beta,N} = \log N / \log \beta$.

However, when $\beta \in (\beta_c(N), N)$ we know little about the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$. When $N = 2$, Daróczy and Kátai [12] gave a method to calculate the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$ only if β is a purely Parry number. When $N > 2$, Kallós [20] showed that when $\beta \in [N-1, (N-1+\sqrt{N^2-2N+5})/2]$ the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$ is given by $\dim_H \boldsymbol{U}_{\beta,N} = \log(N-2)/\log\beta$. Later in [21] he investigated the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$ for $\beta \in [(N-1+\sqrt{N^2-2N+5})/2, N)$, and gave a method to calculate its Hausdorff dimension when β is a purely Parry number.

In this paper we improve the main results of Kallós [20, 21]. In theorem 2.6 we give the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$ for β in any admissible interval $[\beta_L, \beta_U]$, where β_L is a purely Parry number while β_U is a transcendental number. Moreover, we show in theorem 2.5 that all of these admissible intervals cover almost every point of $(\beta_c(N), N)$. Therefore, we are able to calculate the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$ for almost every $\beta > 1$. In particular, we give explicit formula for the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$ when β is in any one-level or two-level admissible intervals $[\beta_L, \beta_U]$ (see theorems 7.1 and 7.2 for more explanation).

Example 1.1. *Let $N = 10$. By theorem 7.1, theorem 7.2 and the above observation we plot in figure 1 that the graph of the dimension function $f(\beta) = \dim_H \boldsymbol{U}_{\beta,10}$ for $\beta \in (1, 110)$. In particular, we give a detailed plot of $f(\beta)$ for β in the one-level and two-level admissible intervals in $(\beta_c(10), 10) \approx (5.976, 10)$. Clearly, the dimension function $f(\beta)$ fluctuates frequently for $\beta \in (\beta_c(10), 10)$. In [22] we will show that $f(\beta)$ is continuous for $\beta > 1$.*

The structure of the paper is arranged as follows. In section 2 we introduce the admissible blocks, admissible intervals and the generalized Thue–Morse sequences, and state our main results as in theorems 2.3, 2.5 and 2.6. In section 3 we presented some properties of unique beta expansions. The proofs of theorems 2.3, 2.5 and 2.6 are given in sections 4, 5 and 6, respectively. In section 7 we consider some examples for which the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$ can be calculated explicitly.

2. Preliminary and main results

Given an integer $N \geqslant 2$, for any $\beta \in (1, N)$ the set $\Gamma_{\beta,N}$ is a closed interval, i.e. $\Gamma_{\beta,N} = [0, (N-1)/(\beta-1)]$. Then any real number in this interval $\Gamma_{\beta,N}$ has a β-expansion, some of them may have multiple β-expansions. Among these expansions we define the so-called *greedy β-expansion* $(b_i(x)) = (b_i)$ of $x \in \Gamma_{\beta,N}$ recursively as follows (see [29]). For $x \in \Gamma_{\beta,N}$, if b_i has already been defined for $1 \leqslant i < n$ (no condition if $n = 1$), then b_n is the largest element in $\{0, 1, \cdots, N-1\}$ satisfying

$$\frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \cdots + \frac{b_n}{\beta^n} \leqslant x.$$

One can verify that (b_i) is indeed a β-expansion of x. Moreover, (b_i) is the largest β-expansion of x in the sense of lexicographical order among all β-expansions of x.

Accordingly, we define the so-called *quasi-greedy β-expansion* $(a_i(x)) = (a_i)$ of $x \in \Gamma_{\beta,N}$ recursively as follows (see [13]). For $x = 0$ we set $(a_i) = 0^\infty$. For $x \in \Gamma_{\beta,N} \setminus \{0\}$, if a_i has already been defined for $1 \leqslant i < n$ (no condition if $n = 1$), then a_n is the largest element in $\{0, 1, \cdots, N-1\}$ satisfying

$$\frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_n}{\beta^n} < x.$$

One can also verify that (a_i) is indeed a β-expansion of x. Clearly, the quasi-greedy β-expansion of x is the largest infinite β-expansion of x in the sense of lexicographical order among all β-expansions of x. Here we call a β-expansion *infinite* if the expansion has infinitely many non-zero elements.

For a positive integer p let $\{0, 1, \cdots, N-1\}^p$ be the set of blocks $c_1 \cdots c_p$ of length p with each element $c_i \in \{0, 1, \cdots, N-1\}$. For two blocks $c_1 \cdots c_p$ and $d_1 \cdots d_q$ let $c_1 \cdots c_p d_1 \cdots d_q \in \{0, 1, \cdots, N-1\}^{p+q}$ denote their concatenation. In particular, let $(c_1 \cdots c_p)^k$ denote the k times concatenations of $c_1 \cdots c_p$ to itself, and let $(c_1 \cdots c_p)^\infty$ denote the infinite concatenations of $c_1 \cdots c_p$ to itself. For a digit $c \in \{0, 1, \cdots, N-1\}$ its *reflection* $\overline{c}$ is defined by

$$\overline{c} := N - 1 - c.$$

Accordingly, if $c_i \in \{0, 1, \cdots, N-1\}$ for $i \geqslant 1$, we shall also write $\overline{c_1 \cdots c_n}$ instead of $\overline{c_1} \cdots \overline{c_n}$, and $\overline{c_1 c_2 \cdots}$ instead of $\overline{c_1}\,\overline{c_2} \cdots$. Finally, for a block $c_1 \cdots c_p \in \{0, 1, \cdots, N-1\}^p$ with $c_p > 0$ we set

$$c_1 \cdots c_p^- := c_1 \cdots c_{p-1}(c_p - 1).$$

Similarly, for a block $c_1 \cdots c_p \in \{0, 1, \cdots, N-1\}^p$ with $c_p < N-1$ we set

$$c_1 \cdots c_p^+ := c_1 \cdots c_{p-1}(c_p + 1).$$

In particular, when $p = 1$ we set $c_1 \cdots c_p^- = c_1^- = c_1 - 1$ and $c_1 \cdots c_p^+ = c_1^+ = c_1 + 1$.

In the following we will use lexicographical order between blocks and sequences.

Definition 2.1. *A block $t_1 \cdots t_p \in \{0, 1, \cdots, N-1\}^p$ is called an admissible block if $t_p < N-1$ and for any $1 \leqslant i \leqslant p$ we have*

$$\overline{t_1 \cdots t_p} \leqslant t_i \cdots t_p t_1 \cdots t_{i-1} \quad \text{and} \quad t_i \cdots t_p^+ \overline{t_1 \cdots t_{i-1}} \leqslant t_1 \cdots t_p^+.$$

Clearly, there exist infinitely many admissible blocks. In the following we introduce a generalized Thue–Morse sequence which plays an essential role in this paper.

Definition 2.2. *For a block $t_1 \cdots t_p \in \{0, 1, \cdots, N-1\}^p$ with $t_p < N-1$, we call the sequence $(\theta_i) = (\theta_i(t_1 \cdots t_p^+))$ a generalized Thue–Morse sequence generated by the block $t_1 \cdots t_p^+$ if (θ_i) can be defined by induction as follows. First, we set*

$$\theta_1 \cdots \theta_p = t_1 \cdots t_p^+.$$

Then, if $\theta_1 \cdots \theta_{2^m p}$ is already defined for some nonnegative integer m, we set

$$\theta_{2^m p+1} \cdots \theta_{2^{m+1} p} = \overline{\theta_1 \cdots \theta_{2^m p}}^{\,+}.$$

We first discovered the generalized Thue–Morse sequences from the work of De Vries and Komornik [13]. Later, we found that these sequences were previously studied by Allouche and Cosnard [1], Komornik and Loreti [24], *et al*.

If $N = 2k+1$, then the sequence $(\lambda_i(N))$ defined in equation (3) is exactly the generalized Thue–Morse sequence $(\theta_i(k+1))$. If $N = 2k$, using lemma 4.6 one can also show that $(\lambda_i(N)) = (\theta_i(k))$. Thus, for any $N \geqslant 2$ we have

$$(\lambda_i(N)) = \left(\theta_i\left(\left\lceil \frac{N}{2} \right\rceil\right)\right),$$

where $\lceil x \rceil$ denotes the least integer larger than or equal to x.

In the rest of the paper we will reserve the notation $(\alpha_i(\beta))$ especially for the quasi-greedy β-expansion of $1 \in \Gamma_{\beta,N} = [0, (N-1)/(\beta-1)]$ (since $\beta \in (1, N)$).

Theorem 2.3. *For $N \geqslant 2$, let $t_1 \cdots t_p \in \{0, 1, \cdots, N-1\}^p$. Then $t_1 \cdots t_p$ is an admissible block if and only if $(\alpha_i(\beta_L)) = (t_1 \cdots t_p)^\infty$ and $(\alpha_i(\beta_U)) = (\theta_i(t_1 \cdots t_p^+))$ for some bases $\beta_L, \beta_U \in [G_N, N)$, where G_N is the critical base defined in (2). Moreover, $\beta_L < \beta_U$, and β_L is algebraic while β_U is transcendental.*

We point out that theorem 2.3 generalizes some results in [2] and [23]. Here we call the transcendental numbers β_U *De Vries–Komornik constants* since these numbers were first studied by De Vries and Komornik in [13]. Later in proposition 4.3 and theorem 4.4 we will show that $t_1 \cdots t_p$ is an admissible block if and only if $(\alpha_i(\beta_U)) = (\theta_i(t_1 \cdots t_p^+))$, if and only if $(\theta_i(t_1 \cdots t_p^+))$ is the unique β_U-expansion of 1.

Definition 2.4. *The closed interval $[\beta_L, \beta_U]$ given in theorem 2.3 is called an admissible interval generated by $t_1 \cdots t_p$ (simply called, admissible interval) if*

$$(\alpha_i(\beta_L)) = (t_1 \cdots t_p)^\infty \quad \text{and} \quad (\alpha_i(\beta_U)) = (\theta_i(t_1 \cdots t_p^+)).$$

Since we have infinitely many admissible intervals, it is worthwhile to investigate the size of union of these admissible intervals and the relationship between them as well.

Theorem 2.5. *The union of all admissible intervals covers almost every point of $(\beta_c(N), N)$, where $\beta_c(N)$ is the Komornik–Loreti constant. Moreover, for any two admissible intervals $[\alpha_L, \alpha_U]$ and $[\beta_L, \beta_U]$, either $[\alpha_L, \alpha_U] \cap [\beta_L, \beta_U] = \emptyset$ or $\alpha_U = \beta_U$.*

Theorem 2.5 says that for any two admissible intervals, either they are separated from each other or they have the same right endpoint. Now we state our main result on the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$.

Theorem 2.6. *For $N \geqslant 2$, let $[\beta_L, \beta_U]$ be an admissible interval generated by $t_1 \cdots t_p$. Then for any $\beta \in [\beta_L, \beta_U]$ the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$ is given by*

$$\dim_H \boldsymbol{U}_{\beta,N} = \frac{h(Z_{t_1 \cdots t_p})}{\log \beta},$$

where $h(Z_{t_1 \cdots t_p})$ is the topological entropy of the subshift of finite type

$$Z_{t_1 \cdots t_p} := \left\{(d_i) : \overline{t_1 \cdots t_p} \leqslant d_n \cdots d_{n+p-1} \leqslant t_1 \cdots t_p, n \geqslant 1\right\}.$$

We point out that when $N = 2$ Barrera [8] investigated the topological entropy of $\mathbf{U}_{\beta,N}$. We also point out that theorem 2.6 generalizes some results in [5,12,20,21]. This will be explained in section 7 via some examples for which the Hausdorff dimension of $\mathbf{U}_{\beta,N}$ can be calculated explicitly.

3. Properties of unique expansions

Recall that $(\alpha_i(\beta))$ is the quasi-greedy β-expansion of 1. The following characterization of $(\alpha_i(\beta))$ can be proved by a slight modification of the proof of [13, proposition 2.3] (see also, [6, theorem 2.2]).

Proposition 3.1. *Let $N \geqslant 2$ and $(\alpha_i(\beta))$ be the quasi-greedy β-expansion of* 1 *w.r.t. the digit set* $\{0, 1, \cdots, N-1\}$. *Then the map $\beta \to (\alpha_i(\beta))$ is a strictly increasing bijection from the interval* $(1, N]$ *onto the set of all infinite sequences $(\gamma_i) \in \{0, 1, \cdots, N-1\}^\infty$ satisfying*

$$\gamma_{k+1}\gamma_{k+2}\cdots \leqslant \gamma_1\gamma_2\cdots \quad \text{for all } k \geqslant 0.$$

Moreover, the map $\beta \to (\alpha_i(\beta))$ is continuous w.r.t. the topology in $\{0, 1, \cdots, N-1\}^\infty$ *induced by the metric defined by* $d\big((c_i), (d_i)\big) = 2^{-\min\{j: c_j \neq d_j\}}$.

In the following we will write (α_i) instead of $(\alpha_i(\beta))$ for the quasi-greedy β-expansion of 1 if no confusion arises for β. The following proposition for the characterization of greedy expansions can be proved in a similar way as in [15] (see also, [6, theorem 3.2]).

Proposition 3.2. *For $N \geqslant 2$ and $\beta \in (1, N]$, $(b_i(x)) = (b_i)$ is the greedy β-expansion of some $x \in [0, (N-1)/(\beta-1)]$ if and only if*

$$b_{n+1}b_{n+2}\cdots < \alpha_1\alpha_2\cdots$$

whenever $b_n < N-1$.

By proposition 3.2 we have an equivalent characterization for the greedy expansions (see also [6, 13]).

Proposition 3.3. *For $N \geqslant 2$ and $\beta \in (1, N]$, $(b_i) = (b_i(x))$ is the greedy β-expansion of some $x \in [0, (N-1)/(\beta-1)]$ if and only if*

$$b_{n+k+1}b_{n+k+2}\cdots < \alpha_1\alpha_2\cdots \tag{4}$$

for all $k \geqslant 0$ whenever $b_n < N-1$.

Proof. The sufficiency follows directly by taking $k = 0$ in equation (4) and then using proposition 3.2. For the necessity, suppose (b_i) is the greedy expansion of some x, and suppose $b_n < N-1$ for some $n \geqslant 1$. By proposition 3.2 we have

$$b_{n+1}b_{n+2}\cdots < \alpha_1\alpha_2\cdots. \tag{5}$$

We claim that $b_{n+2}b_{n+3}\cdots < \alpha_1\alpha_2\cdots$.

If $b_{n+1} < N-1$, proposition 3.2 yields the claim. If $b_{n+1} = N-1$, equation (5) implies that $\alpha_1 = N-1$ and therefore

$$b_{n+2}b_{n+3}\cdots < \alpha_2\alpha_3\cdots \leqslant \alpha_1\alpha_2\cdots,$$

where the second inequality follows from proposition 3.1.

By induction, we have $b_{n+k+1}b_{n+k+2}\cdots < \alpha_1\alpha_2\cdots$ for all $k \geqslant 0$. □

Note that an expansion $(d_i) = (d_i(x))$ is the unique expansion of $x \in \mathbf{U}_{\beta,N}$ if and only if both (d_i) and $(\overline{d_i})$ are the greedy expansions (see [15]). By using proposition 3.3 we have the following characterization of $\mathbf{U}_{\beta,N}$.

Theorem 3.4. *For $\beta \in (1, N]$, let $(\alpha_i) = (\alpha_i(\beta))$ be the quasi-greedy β-expansion of 1. Then $x \in \boldsymbol{U}_{\beta,N}$ if and only if the β-expansion $(d_i) = (d_i(x))$ of x satisfies*

$$\begin{cases} d_{m+k+1}d_{m+k+2}\cdots < \alpha_1\alpha_2\cdots, \\ \overline{d_{n+k+1}d_{n+k+2}\cdots} < \alpha_1\alpha_2\cdots, \end{cases}$$

for all $k \geqslant 0$, where m is the least integer such that $d_m < N-1$ and n is the least integer such that $d_n > 0$.

In terms of theorem 3.4 we can simplify the calculation of the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$ as described in the following theorem.

Theorem 3.5. *For $N \geqslant 2$ and $\beta \in (1, N]$, let $(\alpha_i) = (\alpha_i(\beta))$. Then we have*

$$\dim_H \boldsymbol{U}_{\beta,N} = \dim_H \boldsymbol{W}_{\beta,N},$$

where

$$\boldsymbol{W}_{\beta,N} := \left\{ \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} : \overline{\alpha_1\alpha_2\cdots} < d_n d_{n+1}\cdots < \alpha_1\alpha_2\cdots,\ n \geqslant 1 \right\}.$$

Proof. Clearly, by theorem 3.4 we have $\boldsymbol{W}_{\beta,N} \subseteq \boldsymbol{U}_{\beta,N}$. In terms of the properties of Hausdorff dimension it suffices to show that

$$\begin{aligned} \boldsymbol{U}_{\beta,N} \subseteq & \bigcup_{d=1}^{N-2} \frac{d + \boldsymbol{W}_{\beta,N}}{\beta} \cup \bigcup_{n=1}^{\infty}\bigcup_{d=1}^{N-1} \frac{d + \boldsymbol{W}_{\beta,N}}{\beta^{n+1}} \\ & \cup \bigcup_{m=1}^{\infty}\bigcup_{d=0}^{N-2} \left(\sum_{\ell=1}^{m} \frac{N-1}{\beta^\ell} + \frac{d + \boldsymbol{W}_{\beta,N}}{\beta^{m+1}} \right). \end{aligned} \tag{6}$$

Let $x \in \boldsymbol{U}_{\beta,N}$ and $(d_i) = (d_i(x))$ be its unique β-expansion. We will finish the proof by showing in the following three cases that x is also in the right-hand side of (6).

Case I. $0 < d_1 < N-1$. Then by theorem 3.4 it follows that

$$\overline{\alpha_1\alpha_2\cdots} < d_{k+1}d_{k+2}\cdots < \alpha_1\alpha_2\cdots$$

for all $k \geqslant 1$, i.e., $d_2d_3\cdots \in \Pi_\beta^{-1}(\boldsymbol{W}_{\beta,N})$ where Π_β is the projection map defined in (1).

Case II. $d_1 = 0$. Then by theorem 3.4 it yields that

$$d_{k+1}d_{k+2}\cdots < \alpha_1\alpha_2\cdots$$

for all $k \geqslant 1$. Let n be the least integer such that $d_n > 0$. Again by theorem 3.4 it follows that $d_{n+1}d_{n+2}\cdots \in \Pi_\beta^{-1}(\boldsymbol{W}_{\beta,N})$.

Case III. $d_1 = N-1$. Then in a similar way as in case II we have $d_{m+1}d_{m+2}\cdots \in \Pi_\beta^{-1}(\boldsymbol{W}_{\beta,N})$, where m is the least integer such that $d_m < N-1$. □

Clearly, $\Pi_\beta^{-1}(\boldsymbol{W}_{\beta,N})$ is a symmetric subshift of $\{0, 1, \cdots, N-1\}^\infty$. According to theorem 3.5 it suffices to prove theorem 2.6 for $\boldsymbol{W}_{\beta,N}$ instead of $\boldsymbol{U}_{\beta,N}$.

4. Proof of theorem 2.3

Suppose that $t_1\cdots t_p \in \{0, 1, \cdots, N-1\}^p$ is an admissible block. Then by definition 2.1 it follows

$$\overline{t_1\cdots t_p} \leqslant t_i\cdots t_p t_1\cdots t_{i-1} < t_i\cdots t_p^+ \overline{t_1\cdots t_{i-1}} \leqslant t_1\cdots t_p^+ \tag{7}$$

for any $1 \leqslant i \leqslant p$. The following proposition guarantees that $(t_1\cdots t_p)^\infty$ is a quasi-greedy expansion of 1 for some base $\beta_L \in (1, N]$.

Proposition 4.1. *Let $t_1 \cdots t_p \in \{0, 1, \cdots, N-1\}^p$ be an admissible block. Then $(\alpha_i(\beta_L)) = (t_1 \cdots t_p)^\infty$ for some base $\beta_L \in (1, N]$.*

Proof. Since $t_1 \cdots t_p \in \{0, 1, \cdots, N-1\}^p$ is an admissible block, by (7) it follows that

$$t_i \cdots t_p t_1 \cdots t_{i-1} \leqslant t_1 \cdots t_p$$

for any $1 \leqslant i \leqslant p$. This yields

$$t_i \cdots t_p (t_1 \cdots t_p)^\infty = (t_i \cdots t_p t_1 \cdots t_{i-1})^\infty \leqslant (t_1 \cdots t_p)^\infty.$$

Then by proposition 3.1 we have $(\alpha_i(\beta_L)) = (t_1 \cdots t_p)^\infty$ for some $\beta_L \in (1, N]$. □

Note that $\beta > 1$ is a purely Parry number if $(\alpha_i(\beta))$ is periodic. Hence, the base β_L defined in proposition 4.1 is a purely Parry number. Later in proposition 4.5 we will show that $\beta_L \geqslant G_N$. Recall from definition 2.2 that $(\theta_i) = (\theta_i(t_1 \cdots t_p^+))$ is a generalized Thue–Morse sequence. We will show in proposition 4.3 that if $t_1 \cdots t_p$ is admissible, then (θ_i) is also a quasi-greedy expansion of 1 for some base β_U. First we give the following lemma.

Lemma 4.2. *Let $t_1 \cdots t_p$ be an admissible block and let $(\theta_i) = (\theta_i(t_1 \cdots t_p^+))$ be the generalized Thue–Morse sequence generated by $t_1 \cdots t_p^+$. Then for any $n \geqslant 0$ we have*

$$\overline{\theta_1 \cdots \theta_{2^n p-i+1}} < \theta_i \cdots \theta_{2^n p} \leqslant \theta_1 \cdots \theta_{2^n p-i+1} \tag{8}$$

for any $1 \leqslant i \leqslant 2^n p$.

Proof. We will prove (8) using induction on n. Since $t_1 \cdots t_p$ is an admissible block, it follows from equation (7) that

$$\overline{\theta_1 \cdots \theta_{p-i+1}} \leqslant t_i \cdots t_p < t_i \cdots t_p^+ = \theta_i \cdots \theta_p \leqslant \theta_1 \cdots \theta_{p-i+1}$$

for any $1 \leqslant i \leqslant p$. Then (8) holds for $n = 0$.

Suppose (8) holds for $n = k$. We will split the proof of (8) for $n = k+1$ into the following two cases.

Case I. $1 \leqslant i \leqslant 2^k p$. Then by induction we have $\theta_i \cdots \theta_{2^k p} > \overline{\theta_1 \cdots \theta_{2^k p-i+1}}$, which yields

$$\theta_i \cdots \theta_{2^{k+1} p} > \overline{\theta_1 \cdots \theta_{2^{k+1} p-i+1}}.$$

Again by induction we have $\theta_i \cdots \theta_{2^k p} \leqslant \theta_1 \cdots \theta_{2^k p-i+1}$, and for any $2 \leqslant i \leqslant 2^k p$,

$$\theta_{2^k p+1} \cdots \theta_{2^k p+i-1} = \overline{\theta_1 \cdots \theta_{i-1}} < \theta_{2^k p-i+2} \cdots \theta_{2^k p},$$

where the inequality holds by the induction. Then

$$\overline{\theta_1 \cdots \theta_{2^{k+1} p-i+1}} < \theta_i \cdots \theta_{2^{k+1} p} \leqslant \theta_1 \cdots \theta_{2^{k+1} p-i+1}$$

for any $1 \leqslant i \leqslant 2^k p$.

Case II. $2^k p < i \leqslant 2^{k+1} p$. Then we can write $i = 2^k p + j$ with $1 \leqslant j \leqslant 2^k p$. By induction and definition 2.2 of the generalized Thue–Morse sequence (θ_i) it follows that

$$\overline{\theta_1 \cdots \theta_{2^{k+1} p-i+1}} < \overline{\theta_j \cdots \theta_{2^k p}}^{\,+} = \theta_i \cdots \theta_{2^{k+1} p} \leqslant \theta_1 \cdots \theta_{2^{k+1} p-i+1}$$

for any $2^k p < i = 2^k p + j \leqslant 2^{k+1} p$. □

Proposition 4.3. *The block $t_1 \cdots t_p \in \{0, 1, \cdots, N-1\}^p$ is admissible if and only if $(\alpha_i(\beta_U)) = (\theta_i(t_1 \cdots t_p^+))$ for some base $\beta_U \in (1, N]$.*

Proof. We first prove the sufficiency. Suppose $(\alpha_i(\beta_U)) = (\theta_i(t_1\cdots t_p^+))$ for some $\beta_U \in (1, N]$. By definition 2.2 the generalized Thue–Morse sequence $(\theta_i(t_1\cdots t_p^+))$ begins with

$$(\theta_i(t_1\cdots t_p^+)) = t_1\cdots t_p^+\,\overline{t_1\cdots t_p}\,\overline{t_1\cdots t_p^+}\,t_1\cdots t_p^+\cdots. \tag{9}$$

Then by proposition 3.1 it follows that

$$t_i\cdots t_p^+\overline{t_1\cdots t_{i-1}} \leqslant t_1\cdots t_p^+ \quad \text{and} \quad \overline{t_i\cdots t_p t_1\cdots t_{i-1}} \leqslant t_1\cdots t_p^+$$

for any $1 \leqslant i \leqslant p$. By definition 2.1 it suffices to show that $\overline{t_i\cdots t_p t_1\cdots t_{i-1}} \neq t_1\cdots t_p^+$ for all $1 \leqslant i \leqslant p$.

Suppose $\overline{t_i\cdots t_p t_1\cdots t_{i-1}} = t_1\cdots t_p^+$ for some $1 \leqslant i \leqslant p$. Then by proposition 3.1 and equation (13) it follows that

$$\overline{t_i\cdots t_p^+} \leqslant \overline{t_1\cdots t_{p-i+1}}.$$

Observing by proposition 3.1 that $t_i\cdots t_p^+ \leqslant t_1\cdots t_{p-i+1}$ we obtain

$$\overline{t_i\cdots t_p t_1\cdots t_p^+} = t_1\cdots t_p^+\overline{t_1\cdots t_{p-i+1}}.$$

This implies that $i \neq 1$. Again by proposition 3.1 and equation (13) we obtain

$$t_1\cdots t_{i-1} = \overline{t_{p-i+2}\cdots t_p} \quad \text{for } 2 \leqslant i \leqslant p.$$

This leads to a contradiction with the assumption that $\overline{t_i\cdots t_p t_1\cdots t_{i-1}} = t_1\cdots t_p^+$.

In the following we will show the necessity. Let $i \geqslant 1$. Then $i < 2^n p$ for some large integer $n \geqslant 0$. By lemma 4.2 it follows that

$$\theta_{i+1}\cdots\theta_{2^n p} \leqslant \theta_1\cdots\theta_{2^n p-i} \quad \text{and} \quad \overline{\theta_1\cdots\theta_i} < \theta_{2^n p-i+1}\cdots\theta_{2^n p}.$$

This implies

$$\begin{aligned}\theta_{i+1}\cdots\theta_{2^n p}\theta_{2^n p+1}\cdots\theta_{2^n p+i}\cdots &= \theta_{i+1}\cdots\theta_{2^n p}\overline{\theta_1\cdots\theta_i}\cdots\\ &< \theta_1\cdots\theta_{2^n p-i}\theta_{2^n p-i+1}\cdots\theta_{2^n p}\cdots.\end{aligned}$$

By proposition 3.1 this establishes the proposition. □

Moreover, using lemma 4.2 one can show that (θ_i) is the unique β_U-expansion of 1.

Theorem 4.4. *Let $t_1\cdots t_p \in \{0, 1, \cdots, N-1\}^p$. Then $t_1\cdots t_p$ is an admissible block if and only if the generalized Thue–Morse sequence $(\theta_i) = (\theta_i(t_1\cdots t_p^+))$ is the unique expansion of 1 for some base β_U, i.e.*

$$\overline{\theta_1\theta_2\cdots} < \theta_{i+1}\theta_{i+2}\cdots < \theta_1\theta_2\cdots \quad \text{for any } i \geqslant 1.$$

Recall from (2) that G_N is the generalized golden ratio. We will show that the admissible intervals are all included in $[G_N, N)$. In proposition 5.2 we will show that all of these admissible intervals cover $(\beta_c(N), N)$ a.e., where $\beta_c(N)(> G_N)$ is the Komornik–Loreti constant.

Proposition 4.5. *Let $[\beta_L, \beta_U]$ be an admissible interval generated by $t_1\cdots t_p$. Then $[\beta_L, \beta_U] \subseteq [G_N, N)$.*

Proof. Clearly, by definition 2.2 of the generalized Thue–Morse sequence $(\theta_i) = (\theta_i(t_1\cdots t_p^+))$ it follows that

$$(\alpha_i(\beta_U)) = (\theta_i) < (N-1)^\infty = (\alpha_i(N)).$$

By proposition 3.1 this implies $\beta_U < N$. In the following we will show $\beta_L \geqslant G_N$.

Since $t_1\cdots t_p$ is admissible, it yields that $t_1 \geqslant \overline{t_1} = N-1-t_1$. Then $t_1 \geqslant \lceil (N-1)/2\rceil$. By definition 2.1 of an admissible block one can directly verify that $(t_1\cdots t_p)^\infty \geqslant (t_1\overline{t_1})^\infty$ (see also

[3, proposition 2]). Note by (2) that $(\alpha_i(G_N)) = (t_1 \cdots t_p)^\infty = (\lceil (N-1)/2 \rceil \overline{\lceil (N-1)/2 \rceil})^\infty$. Then

$$(\alpha_i(\beta_L)) = (t_1 \cdots t_p)^\infty \geqslant (t_1 \overline{t_1})^\infty \geqslant (\alpha_i(G_N)).$$

By proposition 3.1 this implies $\beta_L \geqslant G_N$. □

In the following we will investigate the algebraic properties of the generalized Thue–Morse sequences (θ_i) and show that the De Vries–Komornik constant β_U is transcendental. Recall that $(\tau_i)_{i=0}^\infty$ is the classical Thue–Morse sequence beginning with

$$0110\,1001\,1001\,0110\,1001\,0110\,0110\,1001 \cdots.$$

We write two equivalent definitions for this sequence (τ_i) (see, e.g., [4] for details).

(I) Set $\tau_0 = 0$, $\tau_{2^n} = 1$ for $n = 0, 1, \cdots$, and

$$\tau_{2^n+k} = 1 - \tau_k \quad \text{if} \quad 1 \leqslant k < 2^n, n = 1, 2, \cdots.$$

(II) For a nonnegative integer i we consider its dyadic expansion

$$i = \varepsilon_n 2^n + \varepsilon_{n-1} 2^{n-1} + \cdots + \varepsilon_0, \qquad \varepsilon_k \in \{0, 1\}.$$

Then we set

$$\tau_i = \begin{cases} 0 & \text{if } \sum_{j=0}^n \varepsilon_j \text{ is even,} \\ 1 & \text{if } \sum_{j=0}^n \varepsilon_j \text{ is odd.} \end{cases}$$

Based on (τ_i) we give an equivalent definition for the generalized Thue–Morse sequence (θ_i).

Lemma 4.6. *Let $(\theta_i) = (\theta_i(t_1 \cdots t_p^+))$ be the generalized Thue–Morse sequence generated by $t_1 \cdots t_p^+$. Then for any integer $\ell = ip + q$ with $i \geqslant 0$, $1 \leqslant q \leqslant p$ we have*

$$\theta_\ell = \begin{cases} t_q + \tau_i(\overline{t_q} - t_q), & \text{if} \quad 1 \leqslant q < p \\ t_q + \tau_i(\overline{t_q} - t_q) + (\tau_{i+1} - \tau_i), & \text{if} \quad q = p. \end{cases} \tag{10}$$

Proof. Recall from definition 2.2 that (η_i) is the generalized Thue–Morse sequence generated by $t_1 \cdots t_p^+$ if and only if $\eta_1 \cdots \eta_p = t_1 \cdots t_p^+$, and for any $n \geqslant 0$ we have

$$\eta_{2^{n+1}p} = \overline{\eta_{2^n p}} + 1 = N - \eta_{2^n p}, \qquad \eta_{2^n p + k} = \overline{\eta_k} \quad \text{for all } 1 \leqslant k < 2^n p. \tag{11}$$

Clearly, using $\tau_0 = 0, \tau_1 = 1$ in equation (10) it yields that $\theta_1 \cdots \theta_p = t_1 \cdots t_p^+$. Then it suffices to show that the sequence (θ_i) given in equation (10) satisfies the conditions in (11).

For $n \geqslant 0$, using definition (I) of (τ_i) and equation (10) it follows that

$$\begin{aligned} &\theta_{2^{n+1}p} + \theta_{2^n p} \\ &= \big(t_p + \tau_{2^{n+1}-1}(\overline{t_p} - t_p) + (\tau_{2^{n+1}} - \tau_{2^{n+1}-1})\big) \\ &\quad + \big(t_p + \tau_{2^n-1}(\overline{t_p} - t_p) + (\tau_{2^n} - \tau_{2^n-1})\big) \\ &= t_p + (1 - \tau_{2^n-1})(\overline{t_p} - t_p) + (1 - (1 - \tau_{2^n-1})) \\ &\quad + t_p + \tau_{2^n-1}(\overline{t_p} - t_p) + (1 - \tau_{2^n-1}) \\ &= t_p + \overline{t_p} + 1 = N, \end{aligned}$$

i.e., $\theta_{2^{n+1}p} = N - \theta_{2^n p} = \overline{\theta_{2^n p}} + 1$.

For $1 \leqslant k < 2^n p$ we can write $k = (\varepsilon_{n-1}2^{n-1} + \cdots + \varepsilon_1 2^1 + \varepsilon_0)p + q$ with $\varepsilon_{n-1}, \cdots, \varepsilon_0 \in \{0, 1\}$ and $1 \leqslant q \leqslant p$. Without loss of generality we may assume $1 \leqslant q < p$. If $\sum_{j=0}^{n-1} \varepsilon_j$ is even, then using definition (II) of (τ_i) and equation (10) it follows that

$$\theta_k = \theta_{(\sum_{j=0}^{n-1} \varepsilon_j 2^j)p+q} = t_q + 0(\overline{t_q} - t_q) = t_q,$$

and

$$\theta_{2^n p+k} = \theta_{(2^n+\sum_{j=0}^{n-1}\varepsilon_j 2^j)p+q} = t_q + 1(\overline{t_q} - t_q) = \overline{t_q}.$$

Hence, $\theta_{2^n p+k} = \overline{\theta_k}$. Similarly, if $\sum_{j=0}^{n-1}\varepsilon_j$ is odd , one can also show that $\theta_{2^n p+k} = \overline{\theta_k}$. □

The following theorem for transcendental numbers is due to Mahler [27] (see also [23]).

Theorem 4.7 (Mahler [27]). *If z is an algebraic number in the open unit disc, then the number*

$$Z := \sum_{i=1}^{\infty} \tau_i z^i$$

is transcendental, where (τ_i) is the classical Thue–Morse sequence.

Proof of theorem 2.3. Clearly, by proposition 3.1 $\beta_L < \beta_U$. By propositions 4.1, 4.3 and 4.5 it remains to show that the De Vries–Komornik constant β_U is transcendental.

Let $(\theta_\ell) = (\theta_\ell(t_1 \cdots t_p^+))$ be the generalized Thue–Morse sequence generated by the block $t_1 \cdots t_p^+$. By the definition of β_U we have

$$1 = \sum_{\ell=1}^{\infty} \theta_\ell {\beta_U}^{-\ell}.$$

For any integer $\ell \geqslant 1$, let $\ell = ip + q$ with $i \geqslant 0$ and $1 \leqslant q \leqslant p$. Then using lemma 4.6 we can rewrite the above equation as follows.

$$\begin{aligned}
1 &= \sum_{\ell=1}^{\infty} \theta_\ell {\beta_U}^{-\ell} = \sum_{i=0}^{\infty}\sum_{q=1}^{p} \theta_{ip+q} {\beta_U}^{-ip-q} \\
&= \sum_{i=0}^{\infty} {\beta_U}^{-ip}\Big(\sum_{q=1}^{p}\big(t_q + \tau_i(\overline{t_q} - t_q)\big){\beta_U}^{-q} + (\tau_{i+1} - \tau_i){\beta_U}^{-p}\Big) \\
&= \sum_{i=0}^{\infty} {\beta_U}^{-ip}\Big(\sum_{q=1}^{p} t_q {\beta_U}^{-q}\Big) + \sum_{i=0}^{\infty} \tau_i {\beta_U}^{-ip}\Big(\sum_{q=1}^{p}(\overline{t_q} - t_q){\beta_U}^{-q}\Big) \\
&\quad + \sum_{i=0}^{\infty}(\tau_{i+1} - \tau_i){\beta_U}^{-ip-p} \\
&= \frac{\sum_{q=1}^{p} t_q {\beta_U}^{-q}}{1 - {\beta_U}^{-p}} + \Big(\sum_{i=1}^{\infty}\tau_i({\beta_U}^{-p})^i\Big)\Big(\sum_{q=1}^{p}(\overline{t_q} - t_q){\beta_U}^{-q}\Big) \\
&\quad + \sum_{i=1}^{\infty}\tau_i({\beta_U}^{-p})^i - {\beta_U}^{-p}\sum_{i=1}^{\infty}\tau_i({\beta_U}^{-p})^i,
\end{aligned}$$

where the last equality holds since $\tau_0 = 0$. Rearranging the above equation it gives

$$\sum_{i=1}^{\infty}\tau_i({\beta_U}^{-p})^i = \frac{1 - {\beta_U}^{-p} - \sum_{q=1}^{p} t_q {\beta_U}^{-q}}{(1 - {\beta_U}^{-p})\Big(1 - {\beta_U}^{-p} + \sum_{q=1}^{p}(\overline{t_q} - t_q){\beta_U}^{-q}\Big)}.$$

If $\beta_U > 1$ is an algebraic number, then the right-hand side would be algebraic, while the left-hand side would be transcendental by theorem 4.7. This contradiction implies that β_U is transcendental. □

5. Proof of theorem 2.5

First we will show that all of the admissible intervals cover almost every point of $(\beta_c(N), N)$. Let $\boldsymbol{U}$ be the set of $\beta \in (1, N]$ for which $1 \in \Gamma_{\beta,N}$ has a unique β-expansion, i.e. there exists a unique sequence $(d_i) \in \{0, 1, \cdots, N-1\}^\infty$ such that $1 = \sum_{i=1}^{\infty} d_i/\beta^i$. Let $\overline{\boldsymbol{U}}$ be the closure of $\boldsymbol{U}$. The following proposition for $\overline{\boldsymbol{U}}$ was first proved by Komornik and Loreti [24] for $\beta \in [N-1, N]$ and recently proved by Komornik *et al* in [22].

Proposition 5.1. *For $\beta \in (1, N]$ let $(\alpha_i) = (\alpha_i(\beta))$ be the quasi-greedy β-expansion of* 1. *Then $\beta \in \overline{\boldsymbol{U}}$ if and only if*

$$\overline{\alpha_1\alpha_2\cdots} < \alpha_{k+1}\alpha_{k+2}\cdots \leqslant \alpha_1\alpha_2\cdots \quad \text{for all } k \geqslant 0.$$

Moreover, $\overline{\boldsymbol{U}}$ has zero Lebesgue measure.

Proposition 5.2. *The union of all admissible intervals covers $(\beta_c(N), N)$ a.e..*

Proof. By proposition 5.1 it suffices to show that $(\beta_c(N), N)$ is covered by $\overline{\boldsymbol{U}}$ and the union of all admissible intervals. Take $\beta \in (\beta_c(N), N)$, and let $(\alpha_i) = (\alpha_i(\beta))$ be the quasi-greedy β-expansion of 1. By proposition 3.1 it gives

$$\alpha_{k+1}\alpha_{k+2}\cdots \leqslant \alpha_1\alpha_2\cdots \quad \text{for any } k \geqslant 0. \tag{12}$$

Suppose $\beta \notin \overline{\boldsymbol{U}}$. By proposition 5.1 it follows that there exists $q \geqslant 0$ such that

$$\alpha_{q+1}\alpha_{q+2}\cdots \leqslant \overline{\alpha_1\alpha_2\cdots}. \tag{13}$$

Let m be the least integer q satisfying (13). Since $\beta > \beta_c(N)$, by proposition 3.1 we have $\alpha_1 > \overline{\alpha_1}$. Then $m \geqslant 1$, and one can verify that $\alpha_m > 0$. We will finish the proof by showing that β is contained in the admissible interval $[\beta_L, \beta_U]$ generated by $\alpha_1\cdots\alpha_m^-$.

First we will show the admissibility of $\alpha_1\cdots\alpha_m^-$. Since $\beta > \beta_c(N)$, by proposition 3.1 it follows that either $\alpha_1^- \geqslant \overline{\alpha_1^-}$ or $\alpha_2 \geqslant k = \alpha_1 > \overline{\alpha_1}$ with $N = 2k$. If $m = 1$, then by the definition of m it gives that $\alpha_1^- \geqslant \overline{\alpha_1^-}$. This yields the admissibility of α_1^-. In the following we will assume $m \geqslant 2$. Since m is the least integer satisfying (13), it follows that

$$\alpha_i\cdots\alpha_m \geqslant \overline{\alpha_1\cdots\alpha_{m-i+1}} \quad \text{for any } 1 \leqslant i \leqslant m.$$

We claim that $\alpha_i\cdots\alpha_m > \overline{\alpha_1\cdots\alpha_{m-i+1}}$ for any $1 \leqslant i \leqslant m$.

Suppose $\alpha_i\cdots\alpha_m = \overline{\alpha_1\cdots\alpha_{m-i+1}}$ for some $1 \leqslant i \leqslant m$. Then by the minimality of m and (12) we have

$$\alpha_{m+1}\alpha_{m+2}\cdots > \overline{\alpha_{m-i+2}\alpha_{m-i+3}\cdots} \geqslant \overline{\alpha_1\alpha_2\cdots},$$

leading to a contradiction with (13).

Hence, $\alpha_i\cdots\alpha_m > \overline{\alpha_1\cdots\alpha_{m-i+1}}$ for any $1 \leqslant i \leqslant m$. This, together with (12), implies that

$$\overline{\alpha_1\cdots\alpha_m^-} \leqslant \alpha_i\cdots\alpha_m^-\alpha_1\cdots\alpha_{i-1} \quad \text{and} \quad \alpha_i\cdots\alpha_m\overline{\alpha_1\cdots\alpha_{i-1}} \leqslant \alpha_1\cdots\alpha_m,$$

for any $1 \leqslant i \leqslant m$. By definition 2.1 $\alpha_1\cdots\alpha_m^-$ is admissible.

Now we will show that $\beta \in [\beta_L, \beta_U]$ with $(\alpha_i(\beta_L)) = (\alpha_1\cdots\alpha_m^-)^\infty$ and $(\alpha_i(\beta_U)) = (\theta_i(\alpha_1\cdots\alpha_m))$. This can be verified using proposition 3.1 in the following equation.

$$(\alpha_1\cdots\alpha_m^-)^\infty < \alpha_1\alpha_2\cdots < \alpha_1\cdots\alpha_m\,\overline{\alpha_1\cdots\alpha_m}^+\cdots = (\theta_i(\alpha_1\cdots\alpha_m)),$$

where the second inequality follows by (13). □

By theorem 2.6 and the proof of proposition 5.2 we are able to calculate the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$ for any $\beta \in (\beta_c(N), N) \setminus \overline{\boldsymbol{U}}$.

Corollary 5.3. *For $\beta \in (\beta_c(N), N) \setminus \overline{\mathcal{U}}$, let $(\alpha_i) = (\alpha_i(\beta))$ and let m be the least integer satisfying*

$$\alpha_{m+1}\alpha_{m+2}\cdots \leqslant \overline{\alpha_1\alpha_2\cdots}.$$

Then $\dim_H \mathbf{U}_{\beta,N} = h(Z_{\alpha_1\cdots\alpha_m^-})/\log\beta$*, where* $h(Z_{\alpha_1\cdots\alpha_m^-})$ *is the topological entropy of*

$$Z_{\alpha_1\cdots\alpha_m^-} = \{(d_i) : \overline{\alpha_1\cdots\alpha_m^-} \leqslant d_n\cdots d_{n+m-1} \leqslant \alpha_1\cdots\alpha_m^-, n \geqslant 1\}.$$

In the following we will investigate the relationship between any two admissible intervals. Let $[\alpha_L, \alpha_U]$ and $[\beta_L, \beta_U]$ be two admissible intervals generated by $s_1\cdots s_q$ and $t_1\cdots t_p$, respectively. Then by definition 2.4

$$(\alpha_i(\alpha_L)) = (s_1\cdots s_q)^\infty, \quad (\alpha_i(\alpha_U)) = (\theta_i(s_1\cdots s_q^+)),$$

and

$$(\alpha_i(\beta_L)) = (t_1\cdots t_p)^\infty, \quad (\alpha_i(\beta_U)) = (\theta_i(t_1\cdots t_p^+)).$$

We will prove that $\alpha_L < \beta_U$ implies $\alpha_U \leqslant \beta_U$. By proposition 3.1 this is equivalent to showing

$$(s_1\cdots s_q)^\infty < (\theta_i(t_1\cdots t_p^+)) \quad \Longrightarrow \quad (\theta_i(s_1\cdots s_q^+)) \leqslant (\theta_i(t_1\cdots t_p^+)). \tag{14}$$

We will split the proof of (14) into the following two cases: *Case I.* $1 \leqslant q < p$ (see lemma 5.5). *Case II.* $q \geqslant p$ (see lemma 5.6). First we give the following lemma.

Lemma 5.4. *Let $t_1\cdots t_p$ be an admissible block. Then for any $q < p/2$ we have $\overline{t_1\cdots t_q}^+ \leqslant t_{q+1}\cdots t_{2q}$.*

Proof. Suppose $\overline{t_1\cdots t_q}^+ > t_{q+1}\cdots t_{2q}$ for some $q < p/2$. Write $p = m2q + j$ with $m \geqslant 1$ and $0 < j \leqslant 2q$. Since $t_1\cdots t_p$ is an admissible block, by (7) it yields that $t_{q+1}\cdots t_{2q} \geqslant \overline{t_1\cdots t_q}$. Hence,

$$t_1\cdots t_{2q} = t_1\cdots t_q\,\overline{t_1\cdots t_q}.$$

Again by (7) it follows that

$$t_{q+1}\cdots t_{3q} = \overline{t_1\cdots t_q}\,t_{2q+1}\cdots t_{3q} \geqslant \overline{t_1\cdots t_{2q}} = \overline{t_1\cdots t_q}t_1\cdots t_q,$$

and $t_{2q+1}\cdots t_{3q} \leqslant t_1\cdots t_q$. This yields $t_{2q+1}\cdots t_{3q} = t_1\cdots t_q$.

By iteration, one can show that

$$t_1\cdots t_p = t_1\cdots t_{m2q+j} = (t_1\cdots t_q\,\overline{t_1\cdots t_q})^m t_1\cdots t_j = (t_1\cdots t_{2q})^m t_1\cdots t_j.$$

This is impossible since otherwise we have by (7) that

$$t_1\cdots t_j^+ = t_{p-j+1}\cdots t_p^+ \leqslant t_1\cdots t_j.$$ □

Lemma 5.5. *Let $s_1\cdots s_q$ and $t_1\cdots t_p$ be two admissible blocks with $1 \leqslant q < p$. If $(s_1\cdots s_q)^\infty < (\theta_i(t_1\cdots t_p^+))$, then $(\theta_i(s_1\cdots s_q^+)) \leqslant (\theta_i(t_1\cdots t_p^+))$.*

Proof. Suppose

$$(s_1\cdots s_q)^\infty < (\theta_i(t_1\cdots t_p^+)) =: (\eta_i). \tag{15}$$

Then $s_1\cdots s_q \leqslant \eta_1\cdots\eta_q$. We claim that $s_1\cdots s_q < \eta_1\cdots\eta_q$.

If $s_1\cdots s_q = \eta_1\cdots\eta_q$, then by (15) and theorem 4.4 it follows that

$$s_1\cdots s_q \leqslant \eta_{q+1}\cdots\eta_{2q} \leqslant \eta_1\cdots\eta_q = s_1\cdots s_q.$$

This yields $\eta_1\cdots\eta_{2q} = (s_1\cdots s_q)^2$. By iteration, we have $(\eta_i) = (s_1\cdots s_q)^\infty$, leading to a contradiction with (15).

Hence, $s_1\cdots s_q < \eta_1\cdots\eta_q$, i.e., $s_1\cdots s_q^+ \leqslant \eta_1\cdots\eta_q$. Set $(\xi_i) := (\theta_i(s_1\cdots s_q^+))$. Clearly, if $\xi_1\cdots\xi_q = s_1\cdots s_q^+ < \eta_1\cdots\eta_q$, then $(\xi_i) < (\eta_i)$. Now we assume

$$\xi_1\cdots\xi_q = \eta_1\cdots\eta_q \quad \text{and} \quad p = 2^n q + j$$

with $n \geqslant 0$ and $0 < j \leqslant 2^n q$. We will split the proof of $(\xi_i) \leqslant (\eta_i)$ into the following two cases.

Case I. $n = 0$. Then $p = q + j$ for $0 < j \leqslant q$. By definition 2.2 and lemma 4.2 it follows that for $0 < j < q$

$$\xi_{q+1}\cdots\xi_{q+j} = \overline{\xi_1\cdots\xi_j} = \overline{\eta_1\cdots\eta_j} < \eta_{p-j+1}\cdots\eta_p = \eta_{q+1}\cdots\eta_{q+j},$$

and for $j = q$,

$$\xi_{q+1}\cdots\xi_{2q} = \overline{\xi_1\cdots\xi_q}^+ = \overline{\eta_1\cdots\eta_q}^+ \leqslant \eta_{q+1}\cdots\eta_{2q}.$$

Then by definition 2.2 we obtain that $(\xi_i) \leqslant (\eta_i)$.

Case II. $n \geqslant 1$. Then $q < p/2$. By definition 2.2 and lemma 5.4 it follows that

$$\xi_1\cdots\xi_{2q} = \xi_1\cdots\xi_q\overline{\xi_1\cdots\xi_q}^+ = \eta_1\cdots\eta_q\overline{\eta_1\cdots\eta_q}^+ \leqslant \eta_1\cdots\eta_{2q}.$$

If $\xi_1\cdots\xi_{2q} < \eta_1\cdots\eta_{2q}$, then $(\xi_i) < (\eta_i)$. Suppose $\xi_1\cdots\xi_{2q} = \eta_1\cdots\eta_{2q}$. Then by iteration we have

$$\xi_1\cdots\xi_{2^n q} \leqslant \eta_1\cdots\eta_{2^n q}.$$

Clearly, if $\xi_1\cdots\xi_{2^n q} < \eta_1\cdots\eta_{2^n q}$, then $(\xi_i) < (\eta_i)$. Now suppose $\xi_1\cdots\xi_{2^n q} = \eta_1\cdots\eta_{2^n q}$. In a similar way as in case I, one can show by definition 2.2 and lemma'4.2 that $\xi_{2^n q+1}\cdots\xi_{2^n q+j} < \eta_{2^n q+1}\cdots\eta_{2^n q+j}$ if $0 < j < 2^n q$, and $\xi_{2^n q+1}\cdots\xi_{2^{n+1} q} \leqslant \eta_{2^n q+1}\cdots\eta_{2^{n+1} q}$ if $p = 2^{n+1}q$. Then $(\xi_i) \leqslant (\eta_i)$. □

Lemma 5.6. *Let $s_1\cdots s_q$ and $t_1\cdots t_p$ be two admissible blocks with $q \geqslant p$. If $(s_1\cdots s_q)^\infty < (\theta_i(t_1\cdots t_p^+))$, then $(\theta_i(s_1\cdots s_q^+)) \leqslant (\theta_i(t_1\cdots t_p^+))$.*

Proof. Suppose

$$(s_1\cdots s_q)^\infty < (\theta_i(t_1\cdots t_p^+)) = (\eta_i) \quad \text{and} \quad q = 2^n p + j \tag{16}$$

with $n \geqslant 0$ and $0 \leqslant j < 2^n p$. Then $s_1\cdots s_{2^n p} \leqslant \eta_1\cdots\eta_{2^n p}$. If $s_1\cdots s_{2^n p} < \eta_1\cdots\eta_{2^n p}$, then by definition 2.2 it follows that

$$(\theta_i(s_1\cdots s_q^+)) = s_1\cdots s_{2^n p}s_{2^n p+1}\cdots < (\eta_i) = (\theta_i(t_1\cdots t_p^+)).$$

We will finish the proof by showing that $s_1\cdots s_{2^n p} \neq \eta_1\cdots\eta_{2^n p}$.

Suppose $s_1\cdots s_{2^n p} = \eta_1\cdots\eta_{2^n p}$. We claim that

$$s_1\cdots s_q = \eta_1\cdots\eta_q. \tag{17}$$

Clearly, if $j = 0$, i.e. $q = 2^n p$, then (17) holds. Now we assume $0 < j < 2^n p$. By (16) and definition 2.2 it follows that

$$s_{2^n p+1}\cdots s_{2^n p+j} \leqslant \eta_{2^n p+1}\cdots\eta_{2^n p+j} = \overline{\eta_1\cdots\eta_j}.$$

Also by the admissibility of $s_1\cdots s_q$ we have

$$s_{2^n p+1}\cdots s_{2^n p+j} \geqslant \overline{s_1\cdots s_j} = \overline{\eta_1\cdots\eta_j}.$$

Then $s_{2^n p+1}\cdots s_{2^n p+j} = \overline{\eta_1\cdots\eta_j} = \eta_{2^n p+1}\cdots\eta_{2^n p+j}$ which yields equation (17).

Using equation (17) in (16) it follows from theorem 4.4 that

$$s_1\cdots s_q \leqslant \eta_{q+1}\cdots\eta_{2q} \leqslant \eta_1\cdots\eta_q = s_1\cdots s_q.$$

Then $\eta_{q+1}\cdots\eta_{2q}=s_1\cdots s_q$. By iteration, we have

$$(\theta_i(t_1\cdots t_p^+))=(\eta_i)=(s_1\cdots s_q)^\infty,$$

leading to a contradiction with (16). □

In the following we will prove that $\alpha_L>\beta_L$ implies $\alpha_U\geqslant\beta_U$. By proposition 3.1 this is equivalent to showing

$$(s_1\cdots s_q)^\infty>(t_1\cdots t_p)^\infty\quad\Longrightarrow\quad(\theta_i(s_1\cdots s_q^+))\geqslant(\theta_i(t_1\cdots t_p^+)).\tag{18}$$

The proof of (18) will also be split into the following two cases. *Case I.* $1\leqslant q<p$ (see lemma 5.7). *Case II.* $q\geqslant p$ (see lemma 5.9).

Lemma 5.7. *Let $s_1\cdots s_q$ and $t_1\cdots t_p$ be two admissible blocks with $1\leqslant q<p$. If $(s_1\cdots s_q)^\infty>(t_1\cdots t_p)^\infty$, then $(\theta_i(s_1\cdots s_q^+))\geqslant(\theta_i(t_1\cdots t_p^+))$.*

Proof. Suppose

$$(s_1\cdots s_q)^\infty>(t_1\cdots t_p)^\infty\quad\text{and}\quad p=nq+j\tag{19}$$

with $n\geqslant 1$ and $1\leqslant j\leqslant q$. Then $(s_1\cdots s_q)^n s_1\cdots s_j\geqslant t_1\cdots t_{nq+j}=t_1\cdots t_p$. If $(s_1\cdots s_q)^n s_1\cdots s_j=t_1\cdots t_p$, then

$$t_{p-j+1}\cdots t_p=s_1\cdots s_j=t_1\cdots t_j,$$

leading to a contradiction with the admissibility of $t_1\cdots t_p$. So, $(s_1\cdots s_q)^n s_1\cdots s_j>t_1\cdots t_p$, i.e., $(s_1\cdots s_q)^n s_1\cdots s_j\geqslant t_1\cdots t_p^+$. Then by definition 2.2 it follows that

$$(\theta_i(s_1\cdots s_q^+))>(s_1\cdots s_q)^n s_1\cdots s_j(N-1)^\infty\geqslant(\theta_i(t_1\cdots t_p^+)).$$ □

When $(s_1\cdots s_q)^\infty>(t_1\cdots t_p)^\infty$ with $q\geqslant p$, it is more involved to prove $(\theta_i(s_1\cdots s_q^+))\geqslant(\theta_i(t_1\cdots t_p^+))$. First we consider the following lemma.

Lemma 5.8. *Let $s_1\cdots s_q$ and $t_1\cdots t_p$ be two admissible blocks with $q\geqslant p$. If $s_1\cdots s_p>t_1\cdots t_p$, then $(\theta_i(s_1\cdots s_q^+))\geqslant(\theta_i(t_1\cdots t_p^+))$.*

Proof. Write $q=2^n p+j$ with $n\geqslant 0$ and $0\leqslant j<2^n p$. Suppose $s_1\cdots s_p>t_1\cdots t_p$, i.e.

$$s_1\cdots s_p\geqslant t_1\cdots t_p^+=(\theta_i(t_1\cdots t_p^+))_{i=1}^p.$$

Clearly, if $s_1\cdots s_p>t_1\cdots t_p^+$, then by definition 2.2 it yields $(\theta_i(s_1\cdots s_q^+))\geqslant(\theta_i(t_1\cdots t_p^+))$. Now we assume $s_1\cdots s_p=t_1\cdots t_p^+$, and split the proof into the following three cases.

Case I. $p\leqslant q<2p$. Then by the admissibility of $s_1\cdots s_q=s_1\cdots s_{p+j}$ it follows that

$$\begin{aligned}(\theta_i(s_1\cdots s_q^+))&=s_1\cdots s_p s_{p+1}\cdots s_{p+j}^+\cdots\\&>s_1\cdots s_p\overline{s_1\cdots s_j}(N-1)^\infty\\&=t_1\cdots t_p^+\overline{t_1\cdots t_j}(N-1)^\infty\geqslant(\theta_i(t_1\cdots t_p^+)).\end{aligned}$$

Case II. $q=2p$. Again by the admissibility of $s_1\cdots s_q$ we have

$$s_1\cdots s_q^+=s_1\cdots s_{2p}^+\geqslant s_1\cdots s_p\overline{s_1\cdots s_p}^+=t_1\cdots t_p^+\overline{t_1\cdots t_p}.$$

This implies that $(\theta_i(s_1\cdots s_q^+))\geqslant(\theta_i(t_1\cdots t_p^+\overline{t_1\cdots t_p}))=(\theta_i(t_1\cdots t_p^+))$.

Case III. $q>2p$. Then by the admissibility of $s_1\cdots s_q$ it follows that

$$s_{p+1}\cdots s_{2p}\geqslant\overline{s_1\cdots s_p}=\overline{t_1\cdots t_p^+}.\tag{20}$$

We claim that the inequality in (20) is strict. Otherwise, by the admissibility of $s_1\cdots s_q$ we have

$$\overline{s_1\cdots s_p}s_{2p+1}\cdots s_{3p}=s_{p+1}\cdots s_{3p}\geqslant\overline{s_1\cdots s_{2p}}=\overline{s_1\cdots s_p}s_1\cdots s_p,$$

and $s_{2p+1}\cdots s_{3p}\leqslant s_1\cdots s_p$. This implies that $s_{2p+1}\cdots s_{3p}=s_1\cdots s_p$. By iteration, we have for $q=2kp+\ell$ with $0<\ell\leqslant 2p$,

$$s_1\cdots s_q=(s_1\cdots s_p\overline{s_1\cdots s_p})^k s_1\cdots s_\ell.$$

Then, $s_{q-\ell+1}\cdots s_q=s_1\cdots s_\ell$, leading to a contradiction with the admissibility of $s_1\cdots s_q$. So, the inequality in (20) is strict, i.e.

$$s_1\cdots s_{2p}\geqslant s_1\cdots s_p\overline{s_1\cdots s_p}^{\,+}=t_1\cdots t_p^+\overline{t_1\cdots t_p}=(\theta_i(t_1\cdots t_p^+))_{i=1}^{2p}.$$

Then, by induction, it follows that $s_1\cdots s_{2^n p}\geqslant(\theta_i(t_1\cdots t_p^+))_{i=1}^{2^n p}$. Again by the same argument as in case I we can show that $(\theta_i(s_1\cdots s_q^+))\geqslant(\theta_i(t_1\cdots t_p^+))$. □

Lemma 5.9. *Let $s_1\cdots s_q$ and $t_1\cdots t_p$ be two admissible blocks with $q\geqslant p$. If $(s_1\cdots s_q)^\infty>(t_1\cdots t_p)^\infty$, then $(\theta_i(s_1\cdots s_q^+))\geqslant(\theta_i(t_1\cdots t_p^+))$.*

Proof. Let $q=np+j$ with $n\geqslant 1$ and $0\leqslant j<p$. Suppose

$$(s_1\cdots s_q)^\infty>(t_1\cdots t_p)^\infty. \tag{21}$$

Then $s_1\cdots s_p\geqslant t_1\cdots t_p$. By lemma 5.8 it suffices to show that $s_1\cdots s_p\neq t_1\cdots t_p$.

Suppose $s_1\cdots s_p=t_1\cdots t_p$. Then by (21) and the admissibility of $s_1\cdots s_q$ it gives that

$$s_1\cdots s_p\geqslant s_{p+1}\cdots s_{2p}\geqslant t_1\cdots t_p=s_1\cdots s_p.$$

Then $s_1\cdots s_{2p}=(t_1\cdots t_p)^2$. By iteration, we have

$$s_1\cdots s_{np}=(t_1\cdots t_p)^n. \tag{22}$$

If $j=0$, i.e. $q=np$, then (22) violates (21). If $0<j<p$, then (22) also leads to a contradiction, since by (21) and the admissibility of $s_1\cdots s_q$ it follows that

$$s_1\cdots s_j>s_{np+1}\cdots s_{np+j}\geqslant t_1\cdots t_j=s_1\cdots s_j.$$ □

Proof of theorem 2.5. By proposition 5.2 it suffices to show that either $[\alpha_L,\alpha_U]\cap[\beta_L,\beta_U]=\emptyset$ or $\alpha_U=\beta_U$. By symmetry it suffices to show that $\alpha_L\in[\beta_L,\beta_U]$ implies $\alpha_U=\beta_U$. This can be verified by the following observations. By lemmas 5.5, 5.6 and proposition 3.1 it follows that

$$\alpha_L<\beta_U\quad\Longrightarrow\quad\alpha_U\leqslant\beta_U.$$

Moreover, by lemmas 5.7, 5.9 and proposition 3.1 it follows that

$$\alpha_L\geqslant\beta_L\quad\Longrightarrow\quad\alpha_U\geqslant\beta_U.$$ □

6. Proof of theorem 2.6

Let $[\beta_L,\beta_U]\subseteq[G_N,N)$ be an admissible interval generated by $t_1\cdots t_p$, i.e.

$$(\alpha_i(\beta_L))=(t_1\cdots t_p)^\infty\quad\text{and}\quad(\alpha_i(\beta_U))=(\theta_i(t_1\cdots t_p^+)).$$

Using lemma 4.2 one can easily get the following lemma.

Lemma 6.1. *Let $t_1\cdots t_p$ be an admissible block and let $(\theta_i)=(\theta_i(t_1\cdots t_p^+))$. Then for any $n\geqslant 0$,*

$$\sigma^i((\theta_1\cdots\theta_{2^n p}\overline{\theta_1\cdots\theta_{2^n p}})^\infty)\leqslant(\theta_1\cdots\theta_{2^n p}\overline{\theta_1\cdots\theta_{2^n p}})^\infty$$

for any $i\geqslant 1$, where σ is the left shift such that $\sigma((a_i))=(a_{i+1})$.

By lemma 6.1 and proposition 3.1 it follows that $(\theta_1\cdots\theta_{2^{n-1}p}\,\overline{\theta_1\cdots\theta_{2^{n-1}p}})^\infty$ is the quasi-greedy expansion of 1 for some base $\beta_n\in(1,N]$, i.e.

$$(\alpha_i(\beta_n))=(\theta_1\cdots\theta_{2^{n-1}p}\,\overline{\theta_1\cdots\theta_{2^{n-1}p}})^\infty. \tag{23}$$

Clearly, $(\alpha_i(\beta_1))=(\theta_1\cdots\theta_p\,\overline{\theta_1\cdots\theta_p})^\infty=(t_1\cdots t_p^+\,\overline{t_1\cdots t_p^+})^\infty$, and the first $2^{n-1}p$ elements of $(\alpha_i(\beta_n))$ coincide with that of the generalized Thue–Morse sequence $(\theta_i(t_1\cdots t_p^+))$. Hence, as $n\to\infty$ the sequence $(\alpha_i(\beta_n))$ increasingly converges to the generalized Thue–Morse sequence $(\theta_i(t_1\cdots t_p^+))$. By proposition 3.1 it gives that β_n converges to β_U from the left.

Recall from theorem 3.5 that $\boldsymbol{W}_{\beta,N}$ is defined by

$$\boldsymbol{W}_{\beta,N}=\left\{\sum_{i=1}^\infty\frac{d_i}{\beta^i}:\ \overline{\alpha_1\alpha_2\cdots}<d_nd_{n+1}\cdots<\alpha_1\alpha_2\cdots,\ n\geqslant 1\right\}.$$

The following lemma investigates all possible blocks occurring in the β-expansions of points in $\boldsymbol{W}_{\beta,N}$ for $\beta\leqslant\beta_1$.

Lemma 6.2. *Let $t_1\cdots t_p$ be an admissible block and let $(\alpha_i(\beta_1))=(t_1\cdots t_p^+\overline{t_1\cdots t_p^+})^\infty$. If $\beta\leqslant\beta_1$, then $\boldsymbol{W}_{\beta,N}\subseteq\Pi_\beta(Z_{t_1\cdots t_p})$, where*

$$Z_{t_1\cdots t_p}:=\left\{(d_i):\overline{t_1\cdots t_p}\leqslant d_n\cdots d_{n+p-1}\leqslant t_1\cdots t_p,n\geqslant 1\right\}.$$

Proof. Since $\beta\leqslant\beta_1$, it follows from proposition 3.1 that $(\alpha_i(\beta))\leqslant(\alpha_i(\beta_1))=(t_1\cdots t_p^+\overline{t_1\cdots t_p^+})^\infty$. Take $x=\Pi_\beta((d_i))\in\boldsymbol{W}_{\beta,N}$. Then for all $n\geqslant 1$,

$$\overline{(t_1\cdots t_p^+}t_1\cdots t_p^+)^\infty\leqslant\overline{(\alpha_i(\beta))}<d_nd_{n+1}\cdots<(\alpha_i(\beta))\leqslant(t_1\cdots t_p^+\overline{t_1\cdots t_p^+})^\infty. \tag{24}$$

This implies

$$\overline{t_1\cdots t_p^+}\leqslant d_nd_{n+1}\cdots d_{n+p-1}\leqslant t_1\cdots t_p^+.$$

We will finish the proof by showing that the inequalities in the above equation are strict.

Suppose $d_nd_{n+1}\cdots d_{n+p-1}=t_1\cdots t_p^+$. Then by equation (24) it follows that $d_{n+p}d_{n+p+1}\cdots d_{n+2p-1}\leqslant\overline{t_1\cdots t_p^+}$. Again by equation (24) we have $d_{n+p}d_{n+p+1}\cdots d_{n+2p-1}\geqslant\overline{t_1\cdots t_p^+}$. Then

$$d_{n+p}d_{n+p+1}\cdots d_{n+2p-1}=\overline{t_1\cdots t_p^+}.$$

By iteration, we have $d_nd_{n+1}\cdots=(t_1\cdots t_p^+\overline{t_1\cdots t_p^+})^\infty$, leading to a contradiction with (24).

Similarly, one can show that $d_nd_{n+1}\cdots d_{n+p-1}\neq\overline{t_1\cdots t_p^+}$. □

By lemma 6.2 it yields that $\dim_H\boldsymbol{W}_{\beta,N}\leqslant\dim_H\Pi_\beta(Z_{t_1\cdots t_p})$ for $\beta\leqslant\beta_1$. In the following lemma we will show that $\dim_H\boldsymbol{W}_{\beta,N}\geqslant\dim_H\Pi_\beta(Z_{t_1\cdots t_p})$ for $\beta\geqslant\beta_L$.

Lemma 6.3. *Let $t_1\cdots t_p$ be an admissible block and let $(\alpha_i(\beta_L))=(t_1\cdots t_p)^\infty$. If $\beta\geqslant\beta_L$, then $\dim_H\boldsymbol{W}_{\beta,N}\geqslant\dim_H\Pi_\beta(Z_{t_1\cdots t_p})$.*

Proof. By the definition of $Z_{t_1\cdots t_p}$ it follows that $(\overline{t_1\cdots t_p})^\infty$ and $(t_1\cdots t_p)^\infty$ are the least and the largest elements in $Z_{t_1\cdots t_p}$, respectively. Accordingly, let t_* and t^* be, respectively, the least and the largest elements in $\Pi_\beta(Z_{t_1\cdots t_p})$, i.e.

$$t_*=\Pi_\beta((\overline{t_1\cdots t_p})^\infty)=\frac{\sum_{i=1}^p\overline{t_i}\,\beta^{p-i}}{\beta^p-1},\qquad t^*=\Pi_\beta((t_1\cdots t_p)^\infty)=\frac{\sum_{i=1}^p t_i\beta^{p-i}}{\beta^p-1}.$$

Set

$$T=\bigcup_{n\geqslant 0}\left(\left\{\sum_{i=1}^n\frac{d_i}{\beta^i}+\frac{t_*}{\beta^n}:0\leqslant d_i\leqslant N-1\right\}\cup\left\{\sum_{i=1}^n\frac{d_i}{\beta^i}+\frac{t^*}{\beta^n}:0\leqslant d_i\leqslant N-1\right\}\right).$$

Clearly, T is a countable set. Then it suffices to show that $\Pi_\beta(Z_{t_1\cdots t_p}) \setminus T \subseteq \boldsymbol{W}_{\beta,N}$. Take $x = \Pi_\beta((d_i)) \in \Pi_\beta(Z_{t_1\cdots t_p}) \setminus T$. We claim that $d_n d_{n+1} \cdots < \alpha_1(\beta)\alpha_2(\beta)\cdots$ for any $n \geqslant 1$.

Suppose that there exists $n_0 \geqslant 1$ such that $d_{n_0} d_{n_0+1} \cdots \geqslant (\alpha_i(\beta))$. Since $\beta \geqslant \beta_L$, by proposition 3.1 it follows that

$$d_{n_0} d_{n_0+1} \cdots \geqslant (\alpha_i(\beta)) \geqslant (\alpha_i(\beta_L)) = (t_1 \cdots t_p)^\infty.$$

Since $x \notin T$, we have $d_{n_0} d_{n_0+1} \cdots > (t_1 \cdots t_p)^\infty$. Then there exists a nonnegative integer s such that $d_{n_0} d_{n_0+1} \cdots d_{n_0+sp-1} = (t_1 \cdots t_p)^s$ and

$$d_{n_0+sp} d_{n_0+sp+1} \cdots d_{n_0+sp+p-1} > t_1 \cdots t_p,$$

leading to a contradiction with $x \in \Pi_\beta(Z_{t_1\cdots t_p})$. Thus, $d_n d_{n+1} \cdots < (\alpha_i(\beta))$ for any $n \geqslant 1$.

Similarly, one can show that $d_n d_{n+1} \cdots > \overline{(\alpha_i(\beta))}$ for any $n \geqslant 1$. So $x \in \boldsymbol{W}_{\beta,N}$, and we conclude that $\Pi_\beta(Z_{t_1\cdots t_p}) \setminus T \subseteq \boldsymbol{W}_{\beta,N}$. □

In the following we will investigate the structure of $\Pi_\beta(Z_{t_1\cdots t_p})$. If $p = 1$, then $\Pi_\beta(Z_{t_1})$ is a self-similar set whose structure is well-studied (see [19]). Hence, we only need to consider the case for $p \geqslant 2$. Note that $(d_i) \in Z_{t_1\cdots t_p}$ if and only if $d_n d_{n+1} \cdots d_{n+p-1} \notin \mathscr{F}$ for any $n \geqslant 1$, where

$$\mathscr{F} := \left\{c_1 \cdots c_p : c_1 \cdots c_p < \overline{t_1 \cdots t_p} \text{ or } c_1 \cdots c_p > t_1 \cdots t_p\right\}.$$

Then $Z_{t_1\cdots t_p}$ is a $p-1$ step of shift of finite type (see [26]). We construct an edge graph $\mathscr{G} = (G, V, E)$ with the vertices set V defined by

$$V := \left\{u_1 \cdots u_{p-1} : \overline{t_1 \cdots t_{p-1}} \leqslant u_1 \cdots u_{p-1} \leqslant t_1 \cdots t_{p-1}\right\}.$$

For two vertices $\boldsymbol{u} = u_1 \cdots u_{p-1}, \boldsymbol{v} = v_1 \cdots v_{p-1} \in V$, we draw an edge $\boldsymbol{uv} \in E$ from $\boldsymbol{u}$ to $\boldsymbol{v}$ and label it $\ell_{\boldsymbol{uv}} = u_1$ if $u_2 \cdots u_{p-1} = v_1 \cdots v_{p-2}$[3] and $u_1 \cdots u_{p-1} v_{p-1} \notin \mathscr{F}$. One can check that the edge graph $\mathscr{G} = (G, V, E)$ is a representation of $Z_{t_1\cdots t_p}$.

Lemma 6.4. *Let $t_1 \cdots t_p$ be an admissible block with $p \geqslant 2$ and let $(\alpha_i(\beta_L)) = (t_1 \cdots t_p)^\infty$. Then for any $\beta \geqslant \beta_L$ the set $\Pi_\beta(Z_{t_1\cdots t_p})$ is a graph-directed set satisfying the SSC.*

Proof. Let $\mathscr{G} = (G, V, E)$ be the edge graph representing $Z_{t_1\cdots t_p}$. For $\boldsymbol{u} = u_1 \cdots u_{p-1} \in V$, let

$$K_{\boldsymbol{u}} := \left\{\sum_{i=1}^\infty \frac{d_i}{\beta^i} : d_i = u_i, 1 \leqslant i \leqslant p-1, \text{ and } d_n \cdots d_{n+p-1} \notin \mathscr{F}, n \geqslant 1\right\}.$$

For an edge $\boldsymbol{uv} \in E$ with $\boldsymbol{u} = u_1 \cdots u_{p-1}, \boldsymbol{v} = v_1 \cdots v_{p-1} \in V$ we define the map $f_{\boldsymbol{uv}}$ as

$$f_{\boldsymbol{uv}}(x) = \frac{x + \ell_{uv}}{\beta} = \frac{x + u_1}{\beta}. \tag{25}$$

We claim that for any $\boldsymbol{u} \in V$,

$$K_{\boldsymbol{u}} = \bigcup_{\boldsymbol{uv} \in E} f_{\boldsymbol{uv}}(K_{\boldsymbol{v}}). \tag{26}$$

Take $\Pi_\beta((s_i)) \in K_{\boldsymbol{u}}$. Then $s_1 = u_1, \cdots, s_{p-1} = u_{p-1}$; and $\overline{t_1 \cdots t_p} \leqslant s_n \cdots s_{n+p-1} \leqslant t_1 \cdots t_p$ for any $n \geqslant 1$. This implies that

$$\boldsymbol{v} := s_2 \cdots s_p = u_2 \cdots u_{p-1} s_p \in V \quad \text{and} \quad \boldsymbol{uv} \in E.$$

[3] When $p = 2$ this holds automatically.

Then by equation (25) we have

$$\Pi_\beta((s_i)) \in f_{uv}(K_v) = \left\{ \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} : d_i = u_i, 1 \leqslant i \leqslant p-1; d_p = s_p; \right.$$
$$\left. \text{and } \overline{t_1 \cdots t_p} \leqslant d_n \cdots d_{n+p-1} \leqslant t_1 \cdots t_p, n \geqslant 1 \right\}.$$

So, $K_u \subseteq \bigcup_{uv \in E} f_{uv}(K_v)$.

For the other inclusion of equation (26) we take $\Pi_\beta((s_i)) \in \bigcup_{uv\in E} f_{uv}(K_v)$. Then there exist $uv \in E$ with $u = u_1 \cdots u_{p-1}$, $v = v_1 \cdots v_{p-1} \in V$ such that $\Pi_\beta((s_i)) \in f_{uv}(K_v)$. This implies that $s_i = u_i, 1 \leqslant i \leqslant p-1;\ s_p = v_{p-1}$ and

$$\overline{t_1 \cdots t_p} \leqslant s_n \cdots s_{n+p-1} \leqslant t_1 \cdots t_p,\ \ n \geqslant 1.$$

So, $\Pi_\beta((s_i)) \in K_u$ and we conclude that $\bigcup_{uv\in E} f_{uv}(K_v) \subseteq K_u$. Then equation (26) holds.

Similarly, one can check that

$$\Pi_\beta(Z_{t_1\cdots t_p}) = \bigcup_{v\in V} K_v.$$

Hence, $\Pi_\beta(Z_1 \cdots t_p)$ is a graph-directed set generated by the IFS $\{(K_u)_{u\in V}, (f_{uv})_{uv\in E}\}$ (see [28]). We will finish the proof by showing that the IFS $\{(K_u)_{u\in V}, (f_{uv})_{uv\in E}\}$ satisfies the SSC.

Since $\beta \geqslant \beta_L$, it follows from the proof of lemma 6.3 that for any $(d_i) \in Z_{t_1\cdots t_p}$ we have

$$\overline{\alpha_1(\beta)\alpha_2(\beta)\cdots} \leqslant d_n d_{n+1} \cdots \leqslant \alpha_1(\beta)\alpha_2(\beta)\cdots \quad \text{for any } n \geqslant 1.$$

By proposition 3.1 this implies that $\Pi_\beta(Z_{t_1\cdots t_p}) \subseteq [0,1]$. Let $uv, uv' \in E$ with $u = u_1 \cdots u_{p-1}$, $v = v_1 \cdots v_{p-1}$ and $v' = v'_1 \cdots v'_{p-1}$. Suppose $v_{p-1} < v'_{p-1}$. Then

$$\sum_{i=1}^{p-1} \frac{u_i}{\beta^i} + \frac{v_{p-1}}{\beta^p} + \sum_{i=1}^{\infty} \frac{d_i}{\beta^{p+i}} \leqslant \sum_{i=1}^{p-1} \frac{u_i}{\beta^i} + \frac{v_{p-1}+1}{\beta^p}$$
$$< \sum_{i=1}^{p-1} \frac{u_i}{\beta^i} + \frac{v'_{p-1}}{\beta^p} + \sum_{i=1}^{\infty} \frac{d'_i}{\beta^{p+i}}$$

for any $(d_i), (d'_i) \in Z_{t_1\cdots t_p}$. This yields $f_{uv}(K_v) \cap f_{uv'}(K_{v'}) = \emptyset$. □

When $p = 1$ one can easily get the following lemma.

Lemma 6.5. *Let t_1 be an admissible block and let $(\alpha_i(\beta_L)) = t_1^\infty$. Then for any $\beta \geqslant \beta_L$ the set $\Pi_\beta(Z_{t_1})$ is a self-similar set satisfying SSC.*

Now we give the Hausdorff dimension of $U_{\beta,N}$ for $\beta \in [\beta_L, \beta_1]$.

Proposition 6.6. *Let $t_1 \cdots t_p$ be an admissible block and let $(\alpha_i(\beta_L)) = (t_1 \cdots t_p)^\infty$, $(\alpha_i(\beta_1)) = (t_1 \cdots t_p^+ \overline{t_1 \cdots t_p^+})^\infty$. Then for any $\beta \in [\beta_L, \beta_1]$ the Hausdorff dimension of $U_{\beta,N}$ is given by*

$$\dim_H U_{\beta,N} = \frac{h(Z_{t_1\cdots t_p})}{\log \beta},$$

where $h(Z_{t_1\cdots t_p})$ is the topological entropy of the subshift of finite type $Z_{t_1\cdots t_p}$.

Proof. By lemmas 6.2, 6.3 and theorem 3.5 it follows that for any $\beta \in [\beta_L, \beta_1]$,

$$\dim_H U_{\beta,N} = \dim_H \Pi_\beta(Z_{t_1\cdots t_p}).$$

By lemmas 6.4 and 6.5 $\Pi_\beta(Z_1\cdots t_p)$ is a graph-directed set or a self-similar set satisfying the SSC. Then the Hausdorff dimension of $\Pi_\beta(Z_{t_1\cdots t_p})$ can be calculated via the topological entropy of $Z_{t_1\cdots t_p}$ (see [26]), i.e. $\dim_H \Pi_\beta(Z_{t_1\cdots t_p}) = h(Z_{t_1\cdots t_p})/\log\beta$. □

Proof of theorem 2.6. Recall from (23) that β_n is defined by

$$(\alpha_i(\beta_n)) = (\theta_1\cdots\theta_{2^{n-1}p}\overline{\theta_1\cdots\theta_{2^{n-1}p}})^\infty.$$

Note by lemma 4.2 that $t_1\cdots t_p^+\overline{t_1\cdots t_p^+}$ is admissible. Then by proposition 6.6 it follows that for any $\beta\in[\beta_1,\beta_2]$

$$\dim_H \boldsymbol{U}_{\beta,N} = \frac{h(Z_{t_1\cdots t_p^+\overline{t_1\cdots t_p^+}})}{\log\beta}.$$

By taking $\beta=\beta_1$ in the above equation and in proposition 6.6 it follows that $h(Z_{t_1\cdots t_p}) = h(Z_{t_1\cdots t_p^+\overline{t_1\cdots t_p^+}})$. Hence, for any $\beta\in[\beta_L,\beta_2]$ we have $\dim_H \boldsymbol{U}_{\beta,N} = h(Z_{t_1\cdots t_p})/\log\beta$. By induction, we have

$$\dim_H \boldsymbol{U}_{\beta,N} = \frac{h(Z_{t_1\cdots t_p})}{\log\beta}$$

for any $\beta\in[\beta_L,\beta_n]$. Letting $n\to\infty$ we have by proposition 3.1 that $\beta_n\to\beta_U$. The authors in [22] showed that the map $\beta\to\dim_H \boldsymbol{U}_{\beta,N}$ is continuous for $\beta>1$. This establishes theorem 2.6. □

Remark 6.7. Let $\overline{\boldsymbol{U}_{\beta,N}}$ denote the closure of $\boldsymbol{U}_{\beta,N}$. The authors in [22] showed for $\beta>1$ that the set $\boldsymbol{U}_{\beta,N}$ may be not closed, and the set $\overline{\boldsymbol{U}_{\beta,N}}\setminus\boldsymbol{U}_{\beta,N}$ is at most countable. Then for $\beta\in[\beta_L,\beta_U]$,

$$\dim_H \overline{\boldsymbol{U}_{\beta,N}} = \dim_H \boldsymbol{U}_{\beta,N} = \frac{h(Z_{t_1\cdots t_p})}{\log\beta}.$$

7. Explicit formulae for the Hausdorff dimensions of $\boldsymbol{U}_{\beta,N}$

In this section we consider some examples for which the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$ can be calculated explicitly. An admissible interval $[\beta_L,\beta_U]$ is called a *p-level admissible interval* if $[\beta_L,\beta_U]$ can be generated by an admissible block $t_1\cdots t_p$ of length p. First we will consider the case for the one-level admissible intervals.

Theorem 7.1. *Given $N\geqslant 3$, let $[\beta_L,\beta_U]$ be an admissible interval generated by an admissible block $t_1\in\{0,1,\cdots,N-1\}$. Then $\lceil (N-1)/2\rceil\leqslant t_1\leqslant N-2$, and for any $\beta\in[\beta_L,\beta_U]$ the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$ is given by*

$$\dim_H \boldsymbol{U}_{\beta,N} = \frac{\log(2t_1+2-N)}{\log\beta}.$$

Proof. By definition 2.4 it follows that $(\alpha_i(\beta_L)) = t_1^\infty$ and $(\alpha_i(\beta_U)) = (\theta_i(t_1+1))$. Since $t_1\in\{0,1,\cdots,N-1\}$ is an admissible block, by definition 2.1 it gives that $\lceil (N-1)/2\rceil\leqslant t_1\leqslant N-2$. By theorem 2.6 it follows that for any $\beta\in[\beta_L,\beta_U]$ the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$ is given by

$$\dim_H \boldsymbol{U}_{\beta,N} = \frac{h(Z_{t_1})}{\log\beta},$$

where $Z_{t_1} = \{(d_i) : \overline{t_1}\leqslant d_n\leqslant t_1, n\geqslant 1\}$. So, the theorem follows by an easy calculation that $h(Z_{t_1}) = \log(t_1-\overline{t_1}+1) = \log(2t_1+2-N)$. □

If we take $t_1 = N-2$ in theorem 7.1, then we extend the main result of Kallós [20]. This can be seen by the following observation. Clearly, $\beta_L = N-1$. By definition 2.2 of the generalized Thue–Morse sequence $(\theta_i(N-1))$ it follows that

$$(\alpha_i(\beta_U)) = (\theta_i(N-1)) = (N-1)1\,0(N-1)\,0(N-2)\,(N-1)1\cdots$$
$$> ((N-1)0)^\infty = \Big(\alpha_i\big(\frac{N-1+\sqrt{N^2-2N+5}}{2}\big)\Big).$$

By proposition 3.1 this implies that $\beta_U > (N-1+\sqrt{N^2-2N+5})/2$.

Now we consider the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$ for β in any two-level admissible intervals.

Theorem 7.2. *Given $N \geqslant 2$, let $[\beta_L, \beta_U]$ be an admissible interval generated by an admissible block t_1t_2. Then $\lceil (N-1)/2\rceil \leqslant t_1 \leqslant N-1$, $\overline{t_1} \leqslant t_2 < t_1$, and for any $\beta \in [\beta_L, \beta_U]$ the Hausdorff dimension of $\boldsymbol{U}_{\beta,N}$ is given by*

$$\dim_H \boldsymbol{U}_{\beta,N} = \frac{\log(2t_1+1-N+\sqrt{(2t_1+1-N)^2+4(2t_2+2-N)}\,)-\log 2}{\log\beta}.$$

Proof. Since t_1t_2 is an admissible block, by definition 2.1 it follows that

$$\overline{t_1} \leqslant t_1 \leqslant N-1 \quad \text{and} \quad \overline{t_1} \leqslant t_2 < t_1.$$

By theorem 2.6 it suffices to calculate the entropy of $Z_{t_1t_2}$.

Let $\mathscr{G} = \{G, V, E\}$ be an edge graph representing the shift of finite type $Z_{t_1t_2}$, where the vertex set $V = \{\overline{t_1}, \overline{t_1}+1, \cdots, t_1\}$ and the edge set E consists of all edges uv satisfying $\overline{t_1t_2} \leqslant uv \leqslant t_1t_2$ for $u, v \in V$. Note that the entropy of $Z_{t_1t_2}$ can be calculated via the spectral radius of the adjacency matrix A of the edge graph $\mathscr{G}$ (see [26]), where A is of size $(t_1-\overline{t_1}+1)\times(t_1-\overline{t_1}+1)$ given by

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & \cdots & 1\\ 1 & 1 & 1 & \cdots & \cdots & \cdots & 1\\ 1 & 1 & 1 & 1 & \cdots & \cdots & 1\\ \vdots & & & \ddots & & & \vdots\\ 1 & \cdots & \cdots & 1 & 1 & 1 & 1\\ 1 & \cdots & \cdots & \cdots & 1 & 1 & 1\\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Here the total number of zeros on the top and the bottom rows are both equal to $t_1 - \overline{t_2} + 1 = t_2 - \overline{t_1} + 1$. Then

$$h(Z_{t_1t_2}) = \log\frac{(2t_1+1-N)+\sqrt{(2t_1+1-N)^2+4(2t_2+2-N)}}{2}.$$

This completes the proof. □

The authors in [18, 25] showed that $\dim_H \boldsymbol{U}_{\beta,N} = 0$ when $\beta = \beta_c(N)$. This can also be viewed by theorem 7.1 and 7.2.

Corollary 7.3. *Given $N \geqslant 2$, for any $\beta \in [G_N, \beta_c(N)]$ we have $\dim_H \boldsymbol{U}_{\beta,N} = 0$.*

Proof. We split the proof into the following two cases.

Case I. $N = 2k$. By equations (2) and (3) it follows that

$$(\alpha_i(G_N)) = (k(k-1))^\infty \quad \text{and} \quad (\alpha_i(\beta_c(N))) = (\theta_i(kk)).$$

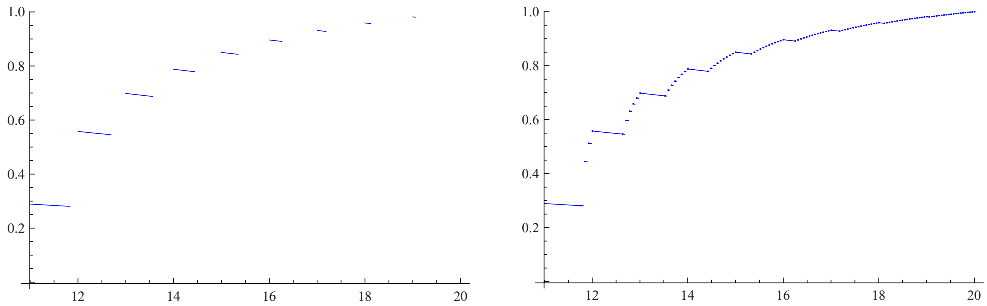

Figure 2. The Hausdorff dimension of $\boldsymbol{U}_{\beta,20}$ for $\beta \in (\beta_c(20), 20)$. In the left column β is in the one-level admissible intervals; in the right column β is in the one-level and two-level admissible intervals.

Hence, $[G_N, \beta_c(N)]$ is an admissible interval generated by the admissible block $k(k-1)$. By theorem 7.2 it follows that for $\beta = \beta_c(N)$ the set $\boldsymbol{U}_{\beta,N}$ has zero Hausdorff dimension.

Case II. $N = 2k + 1$. By equations (2) and (3) one can check that $[G_N, \beta_c(N)]$ is an admissible interval generated by the admissible block k. Then by theorem 7.1 it follows that for $\beta = \beta_c(N)$ we have $\dim_H \boldsymbol{U}_{\beta,N} = 0$. □

Example 7.4. *Let* $N = 20$*. According to theorem 7.1 and theorem 7.2, we plot in figure 2 the graph of the Hausdorff dimension* $\dim_H \boldsymbol{U}_{\beta,20}$ *of* $\boldsymbol{U}_{\beta,20}$ *for* $\beta \in (\beta_c(20), 20)$*. Clearly, the one-level and two-level admissible intervals cover a large part of* $[\beta_c(N), N)$*. By theorem 2.5 the union of all admissible intervals covers almost every point of* $(\beta_c(N), N)$*. Thus, the dimension function* $\dim_H \boldsymbol{U}_{\beta,N}$ *has a devil's-staircase-like behaviour.*

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